Proton - Proton scattering at low energies

The accuracy of proton-proton scattering experiments is comparatively higher, due to the fact that the detection of charged particles is easier than that of neutral particles, since the charged particles are capable of producing visible tracks in a cloud chamber. There are two essential differences between the scattering by neutron-proton system and proton-proton system. (i) In p-p scattering, in addition to the specific nuclear forces, Coulomb repulsive forces are also present. At low energies below 100 Kev, the Coulombian repulsion prevents any close contact between two protons to let the short range nuclear forces to be operative. So at low energies, the p-p scattering predominates due to Coulomb repulsive forces. However as the energy increases beyond this limit the two protons come close enough to allow the nuclear interaction to be effective and thus the interaction potential consists of both the nuclear and Coulombian potential. (ii) Another difference lies in that the scatterer and the scattered particles are identical. This entails some quantum-mechanical complications. The protons being $\frac{1}{2}$ spin particles, are fermions and as such obey Pauli's exclusion principle. As a consequence of this, the wavefunction describing the p-p system must be anti-symmetrical w.r.t the interchange of protons. Therefore, p-p scattering at low energies below 10 Mev involves only the singlet state scattering i.e S-wave scattering and thus the information about nuclear interaction for this system is ~~resis~~ restricted to singlet state only.

The differential scattering cross-section is related to the scattering amplitude $f(\theta)$ calculate in n-p scattering

$$\sigma(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

Since the classical picture of the scattering event envisages distinction between the scatterer particle and (consider) the scattered particle, one would simply add up the cross-section for the two collisions.

The classical differential scattering cross-section therefore is

$$\sigma_{classical}(\theta) = |f(\theta)|^2 \pm |f(\pi-\theta)|^2; \qquad (2)$$

Where $(\pi-\theta)$ appears because the exchange of the protons means changing θ into $(\pi-\theta)$.

But the quantum-mechanical picture does not allow to make a distinction between two identical particles since the waves of the identical particles interfere and the Coulomb forces are repulsive, the interference will mostly by destructive leading to characteristic minima in the differential cross-section at certain angles.

The total wave function must either be symmetrical with total spin $=0$ or antisymmetrical with total spin $=1$ under proton exchange, therefore,

Quantum mechanical differential cross-section is

$$\sigma_{quant}(\theta) = |f(\theta) \pm f(\pi-\theta)|^2$$

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 \pm 2\,Re\left[f(\theta)\, f^*(\pi-\theta)\right] \qquad (3)$$

where in the last term of equation (3) +ve sign refers to the singlet state and -ve sign to the triplet state.

At very low energies, the scattering predominates due to Coulomb interaction. The pure Coulomb scattering cross-section was classically calculated by Rutherford which is

$$d\sigma = \frac{e^4 Z_1^2 Z_2^2}{4\mu^2 v^4 \sin^4(\theta/2)} \cdot 2\pi \sin\theta\, d\theta \; ; \qquad (4)$$

where θ is the scattering angle in the centre of mass system. This formula can be applied to protons by substituting $Z_1 = Z_2 = 1$ and $\mu = \frac{M}{2}$ where M is the proton mass.

Then the Rutherford scattering formula for p-p scattering in the C.M system may be written as:

$$d\sigma = \left(\frac{e^2}{Mv^2}\right)^2 \cdot \left[\frac{1}{\sin^4(\theta/2)} + \frac{1}{\cos^4(\theta/2)}\right] d\Omega \qquad (5)$$

This formula may be transformed into a laboratory or L-system.

If E_0 is the incident proton energy and θ_1 be the scattering angle in the L-system then

$E_0^2 = \cancel{\quad}\ \mu^2 v^4$ and $\sin\theta\, d\theta = 4\sin\theta_1 \cos\theta_1\, d\theta$

∴ Rutherford formula reduces to

$$d\sigma = \left(\frac{e^4}{E_0^2}\right)\left(\frac{1}{\sin^4\theta_1} + \frac{1}{\cos^4\theta_1}\right) 2\pi \cos\theta_1 \sin\theta_1\, d\theta_1 \quad (6)$$

The term $\frac{1}{\cos^4(\theta/2)}$ or $\frac{1}{\cos^4(\theta_1)}$ is added in both (5) and (6) because each proton scattered through an angle θ_1 in the L-system is accompanied by a recoil proton at an angle $(\frac{\pi}{2} - \theta_1)$. These recoil protons are not counted in equation (4). However even at fairly low energies, equation (5) and (6) do not give correct result for P-P scattering.

Since the singlet state occurs one quarter of the time and the triplet state three quarter of the time, the observed cross-section therefore is:

$$d\sigma = \left(\frac{1}{4}\right)\sigma_{singlet}(\theta) + \left(\frac{3}{4}\right)\sigma_{triplet}(\theta)$$

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 - Re[f(\theta) f^*(\pi-\theta)]\, d\Omega \quad (7)$$

This expression is due to Mott. If in this equation, we substitute the amplitude for pure Coulomb scattering $f_{Coul}(\theta)$, we get the so called Mott cross-section for P-P scattering without nuclear effects. The expression for $f_{Coul}(\theta)$ can be written directly as —

$$f_{Coul}(\theta) = \frac{e^2}{Mv^2 \sin^2(\theta/2)} \exp\{-i\eta \log_e \sin^2(\theta/2)\} \quad (8)$$

where $\eta = \frac{e^2}{\hbar v}$ and v = the relative velocity of the two particles.

Substituting the value of equation (8) in (7) we obtain the Mott cross section for pure Coulomb scattering in the C.M system to be

$$\left(\frac{d\sigma}{d\Omega}\right)_{Coul} = \left(\frac{e^2}{Mv^2}\right)^2 \left[\frac{1}{\sin^4(\theta/2)} + \frac{1}{\cos^4(\theta/2)} - \frac{\cos\{\eta \log_e \tan^2(\theta/2)\}}{\cos^2(\theta/2)\sin^2(\theta/2)}\right] \quad (9)$$

A notable feature of this equation is that it remains unchanged if θ is replaced by $\pi - \theta$. This simply expresses the indistinguishability of the two collisions. The first two terms of equation (9) are same as first two terms of equation (5) according to classical theory. The third term in (9) is the quantum mechanical interference term arising from the identity of the incident and target particles.

For non-identical (distinguishable) particles, this interference term is missing and then equation (9) reduces to the classical Rutherford formula. The presence of −ve sign in the third term indicates that protons are fermions. [For bosons such as α-particles, a similar interference term occurs with a +ve sign as was demonstrated by the experiments of Chadwick, Blackett and Champion.] For protons of energy 1 MeV and higher $v > c/20$ and $e^2/\hbar v < 1/7$, so the numerator of the third term i.e. $\cos\left[\eta \log_e \tan^2(\theta/2)\right]$ in equation (9) is nearly unity except for scattering angles not too close to $\frac{\pi}{2}$. ~~Excep~~ Except in these regions equation (9) simplifies to

$$\left(\frac{d\sigma}{d\Omega}\right)_{Coul} = \left(\frac{e^2}{Mv^2}\right)^2 \cdot \left[\frac{1}{\sin^4(\theta/2)} + \frac{1}{\cos^4(\theta/2)} - \frac{1}{\sin^2(\theta/2)\cdot\cos^2(\theta/2)}\right] \quad (10)$$

However experiments of White and Tuve, Heydenberg and Hafstad indicated considerably more protons at 45° than given by equation (10) at proton energies of 1 MeV. This is because of the effect of the nuclear interaction potential which has not taken into consideration.

Effect of nuclear potential

Considering the modifications introduced into the formula (9) by nuclear forces between two protons, [It is quite reasonable to assume that the nuclear potential between two protons has the same characteristics as that between a neutron and a proton since the main difference between a proton and a neutron seems to be the electric charge and the nuclear force does not apparently arise from electric charge. Therefore in P-P scattering at low energy, it is expected that only the $l = 0$ i.e. S-wave scattering process will be affected by the nuclear potentials in the case with n-p scattering].

The presence of Coulomb forces in P-P scattering modifies the scattering amplitude to

$$f(\theta) = \frac{e^2}{Mv^2}\,\frac{\exp\left[-i\eta \log_e \sin^2(\theta/2)\right]}{\sin^2(\theta/2)} + \frac{i}{2k}\left(e^{2i\delta_0} - 1\right) \quad (11)$$

(5) where δ_0 is the nuclear phase shift for $l=0$ or S-wave.
∴ We must correct this equation to account for the identity for the two protons. If spherical harmonic expansion of the nuclear plus Coulomb wave function is considered, then under proton exchange or which is the same as changing $\theta \to \pi-\theta$, the spherical harmonic part changes as follows:

$$P_l[\cos(\pi-\theta)] = [-1]^l(\cos\theta); \qquad (12)$$

So let us split $f(\theta)$ into two parts even (symmetrical) and odd (anti symmetrical) under proton exchange i.e

$$f(\theta) = f_{even}(\theta) + f_{odd}(\theta)$$

$$\left.\begin{aligned} f_{even}(\theta) &= \frac{1}{2}[f(\theta)+f(\pi-\theta)] \\ f_{odd}(\theta) &= \frac{1}{2}[f(\theta)-f(\pi-\theta)] \end{aligned}\right\}; \qquad (13)$$

Even parts of $f(\theta)$ are associated with scattering in the singlet spin state and the odd parts with scattering in triplet spin state. Equation (12) shows that the Legendre Polynomials $P_l(\cos\theta)$ are even for even l and odd for odd l. [Hence the splitting ~~is~~ in (13) is equivalent to separating the terms with even and odd l values in the expansion of the wavefunction.]
Therefore in $f_{even}(\theta)$, components with odd l drop out. Hence

$$f_{even}(\theta) = \frac{e^2}{Mv^2}\left[\frac{\exp\{-i\eta\log_e\sin^2(\theta/2)\}}{\sin^2(\theta/2)} + \frac{\exp\{-i\eta\log_e\cos^2(\theta/2)\}}{\cos^2(\theta/2)}\right] + \frac{i}{k}(e^{2i\delta_0}-1) \qquad (14a)$$

$$f_{odd}(\theta) = \frac{e^2}{Mv^2}\left[\frac{\exp\{-i\eta\log_e\sin^2(\theta/2)\}}{\sin^2(\theta/2)} - \frac{\exp\{-i\eta\log_e\cos^2(\theta/2)\}}{\cos^2(\theta/2)}\right] \qquad (14b)$$

The singlet and the triplet scattering add incoherently. Therefore the total differential scattering cross-section is

$$\frac{d\sigma}{d\Omega} = \left[\frac{3}{4}|f_{odd}(\theta)|^2 + \frac{1}{4}|f_{even}(\theta)|^2\right] \qquad (15)$$

(6) With the help of equations (14) and (15), the differential scattering cross-section becomes,

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{Mu^2}\right)^2 \Bigg[\left(\frac{1}{\sin^4(\theta/2)} + \frac{1}{\cos^4(\theta/2)} - \frac{\cos\{\eta \log_e \tan^2(\theta/2)\}}{\cos^2(\theta/2)\sin^2(\theta/2)} \right) - \frac{2}{\eta}\sin\delta_0 \left(\frac{\cos\{\delta_0 + \eta \log_e \sin^2(\theta/2)\}}{\sin^2(\theta/2)} + \frac{\cos\{\delta_0 + \eta \log_e \cos^2(\theta/2)\}}{\cos^2(\theta/2)} \right) + \left(\frac{4}{\eta^2}\right)\sin^2\delta_0 \Bigg] \quad (16)$$